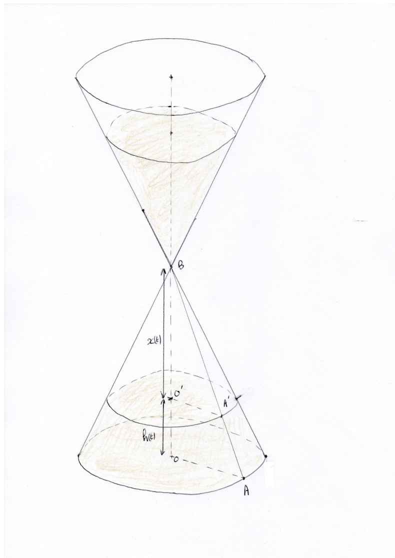
Homework 2

La présentation doit être soignée et toutes les questions doivent être justifiées. La réflexion en groupe est autorisée mais la rédaction des solutions doit être ***personnelle****. La moindre suspicion de recopiage annulera la copie du copieur et du copié.*

Exercise 1. *(7 points).*

Edifis has a limited time to build a palace for the Queen Cleopatra. But he is scatterbrained and forgets the scheduled date. The only information he has, is Cleopatra's hourglass : this one is made of two circular cones connected together at their vertex. The sand falls from the higher cone to the lower cone and will entirely fill the lower cone at the end of the time allowed.

Edifis knows that the radius of the base R is equal to 1 meter and that the height H of the hourglass is equal to 4 meters. Let $x(t)$ be the distance between the vertex of the fallen heap of sand at the time t and the middle of the hourglass and let $h(t)$ be the height of the fallen heap of sand. We denote by O the center of the circle of the base of the hourglass, by A a point of this circle, by B the common vertex of the two cones, by O' the center of the circle created by the top of the fallen heap of sand and by A' the intersection of this circle with (AB).



We define v the function which, at each time t, gives the volume $v(t)$ of the fallen sand. We suppose that the function v is affine. When Edifis came for the first time to see the hourglass, fifteen days ago, at $t = -15$, the height of the sand was

$$h(-15) = 0,5\text{m}.$$

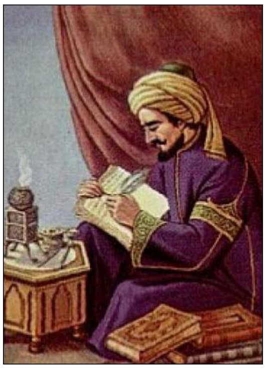
Today, the sand has a height of

$$h(0) = 1\text{m}.$$

1. Let $t \geqslant 0$. Give a formula of $x(t)$ with H and $h(t)$.
2. Give, justifying carefully, a formula of $O'A'$ with R, $x(t)$ and H.
3. Prove that $v(t) = \frac{\pi R^2 H}{6} - \frac{4\pi x(t)^3 R^2}{3H^2}$.
4. Give the numerical values of $v(0)$ and $v(-15)$.
5. Deduce the slope of the function v.
6. Give the y-intercept of v and deduce its algebraic notation.
7. How long does Edifis have to finish his job ?

Exercise 2. *(10 points).* The following text is encoded and we want to reveal the hidden message. We know that the text was encrypted following Cesar's code which consists in moving all the letters with a same number. For example, if this number is 5, then A becomes F, B becomes G, C becomes H and so on.

Ju Trwmr nbc dw bjejwc jajkn md mrg-wndernvn brnlun zdr b'nbc rwcnanbbn j mn wxvkandbnb blrnwlnb juujwc mn uj pnxvncarn j uj vnmnlrwn nc j uj lqrvrn. Mjwb un « Vjwdblarc bda un lqrooanvnwc mnb vnbbjpnb lahycxpajyqrzdnb » ru ngyurzdn lxvvnwc ljbbna unb vnruundab lxmnb lxwwdb j bxw nyxzdn, j u'jrmn mn uj cnlqwrzdn mn u'jwjuhbn mn oanzdnwln. L'nbc uj yanvrnan cajln lxwwdn mn lahycjwjuhbn. Yja lxwbnzdnwc, ru nbc lxwbrmnan lxvvn u'dw mnb oxwmjcndab mn uj mrblryurwn.



The frequency analysis is a statistic approach using the fact that the letters of a language do not appear with the same probability. Indeed, in the French language, the frequencies of each letter are given by the following tabular (in percentage, round at the tenth) :

Rang	1	2	3	4	5	6	7	8	9	10	11	12	13
Lettre	e	a	i	s	t	n	r	u	l	o	d	m	p
Fréquence	15, 87	9, 42	8, 41	7, 9	7, 26	7, 15	6, 46	6, 24	5, 34	5, 14	3, 39	3, 24	2, 86

Rang	14	15	16	17	18	19	20	21	22	23	24	25	26
Lettre	c	v	q	g	b	f	j	h	z	x	y	k	w
Fréquence	2, 64	2, 15	1, 06	1, 04	1, 02	0, 95	0, 89	0, 77	0, 32	0, 3	0, 24	0	0

1. What is the average of the theoretical frequencies ?
2. What is the first quartile and the third quartile ?

We consider that the coded text is a sample of a population of letters satisfying the above frequencies.

3. What criticism can be made on the fact that the letters are independent from each other ?
4. Why cannot we make the prediction interval around the theoretical values given in the above tabular ?

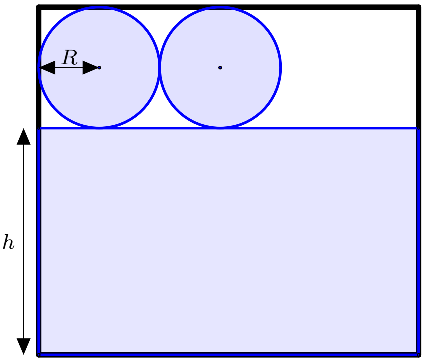
We assume that we can make a prediction interval at level 95% when the following conditions are satisfied :

$$n \geqslant 30 \qquad \text{and} \qquad np > 5 \qquad \text{and} \qquad n(1-p) > 5.$$

5. For which letters can we make a prediction interval at level 95% ?
6. For each letter, make the associated prediction interval at level 95%.
7. What is the letter for which the frequency in the encrypted text is the highest ? Into which prediction interval(s), the frequency of this letter can belong ?
8. Same question of the second letter for which the frequency in the encrypted text is the highest.
9. With the help of question 7, find the number of the shift used to encode the text.
10. Decrypt the message.

Exercise 3. *(7 points).*

A tin can factory wants to optimise its production. The boxes are cylinders with height h and whose disk radium of the base is denoted by R. They are built with metal plate into which we cut the opposite net.
The cost of the raw material is given by the area a of the rectangular plate, bought by the factory. We fix $a = 603$ cm^2. The aim of the factory is to find the better length L and width l in order to obtain the bigger boxes, with a maximum volume.



1. Give a formula of l and L with h and R and deduce a formula of a with h and R.
2. Prove that
$$h = \frac{a}{2\pi R} - 2R.$$
3. Deduce that the volume of the cylinder, made with the above net, is
$$\mathscr{V} = \frac{Ra}{2} - 2\pi R^3.$$
4. We consider $\mathscr{V}$ as a function of the radius R : $\mathscr{V}(R) = 301,5R - 2\pi R^3$ and we assume that this function has a unique maximum when $\mathscr{V}$ is non-negative. More precisely, the variation tabular of $\mathscr{V}$ is given by :

R	0	R_{max}	$\sqrt{\frac{a}{4\pi}}$
$\mathscr{V}$	0 ↗	V_{max}	↘ 0

where R_{max} is the radius for which the volume V_{max} is maximum. We want to approach V_{max} applying the following algorithm.

We start with a radius R equal to 0. This radius increases step by step. Each time, we add to R a small quantity P (for example $P = 0.1$) and we verify if the volume $\mathscr{V}(R+p)$ is bigger or lower than the previous volume $\mathscr{V}(R)$. If the new volume is bigger, our research progresses and we increase R again. Otherwise, we just exceed the maximum. We then stop the algorithm and return the last value of R and of the associated volume. Following this idea, complete the opposite algorithm.

- $R \leftarrow 0$
- Ask P
- $A \leftarrow 301,5 * R - 2 * \pi * R^3$
- $B \leftarrow 301,5 * (R + P) - 2 * \pi * (R + P)^3$
- While $A \ldots\ldots B$
-     $\ldots\ldots \leftarrow R + P$
-     $A \leftarrow B$
-     $B \leftarrow 301,5 * (R+P) - 2 * \pi * (R+P)^3$
- endofwhile
- Return R et A.

5. Construct the algorithm on the calculator and apply it with a step $P = 0.1$. Deduce the value of R_{Max}.
6. With the help of the question 2, deduce h.