Correction de l'exercice de Révision Hiver 09 Analyse asymptotique

Solution de l'exercice 1 On sait que $\mathrm{e}^x \underset{x\to 0}{=} 1+x+\frac{x^2}{2}+\frac{x^3}{6}+o\left(x^3\right)$. De même $\mathrm{e}^{-x} \underset{x\to 0}{=} 1-x+\frac{x^2}{2}-\frac{x^3}{6}+o\left(x^3\right)$. Ainsi,

$$\begin{aligned}
3\,\mathrm{e}^x+\mathrm{e}^{-x} &\underset{x\to 0}{=} 3+3x+3\frac{x^2}{2}+3\frac{x^3}{6}+o\left(x^3\right)+1-x+\frac{x^2}{2}-\frac{x^3}{6}+o\left(x^3\right)\\
&\underset{x\to 0}{=} 4+2x+2x^2+\frac{x^3}{3}+o\left(x^3\right).
\end{aligned}$$

Dès lors, on a

$$f(x) \underset{x\to 0}{=} \ln\left(4+2x+2x^2+\frac{x^3}{3}+o\left(x^3\right)\right) \underset{x\to 0}{=} \ln(4)+\ln\left(1+\frac{x}{2}+\frac{x^2}{2}+\frac{x^3}{12}+o\left(x^3\right)\right).$$

Posons $u(x) \underset{x\to 0}{=} \frac{x}{2}+\frac{x^2}{2}+\frac{x^3}{12}+o\left(x^3\right)$. Alors, $u(x) \underset{x\to 0}{\longrightarrow} 0$. Or $\ln(1+u) \underset{u\to 0}{=} u-\frac{u^2}{2}+\frac{u^3}{3}+o\left(u^3\right)$.

- On a

$$\begin{aligned}
u(x)^2 &\underset{x\to 0}{=} \left(\tfrac{x}{2}+\tfrac{x^2}{2}+\tfrac{x^3}{12}+o\left(x^3\right)\right)\left(\tfrac{x}{2}+\tfrac{x^2}{2}+\tfrac{x^3}{12}+o\left(x^3\right)\right)\\
&\underset{x\to 0}{=} \tfrac{x^2}{4} + \tfrac{x^3}{4} + o\left(x^3\right)\\
&\qquad\quad + \tfrac{x^3}{4} + o\left(x^3\right)\\
&\qquad\quad + o\left(x^3\right)\\
&\underset{x\to 0}{=} \tfrac{x^2}{4}+\tfrac{x^3}{2}+o\left(x^3\right)
\end{aligned}$$

- De plus, $u(x) \underset{x\to 0}{\sim} \frac{x}{2}$ donc par élévation à la puissance,

$$u(x)^3 \underset{x\to 0}{\sim} \frac{x^3}{8} \qquad \text{i.e.} \qquad u(x)^3 \underset{x\to 0}{=} \frac{x^3}{8}+o\left(x^3\right).$$

- Enfin, $o\left(u(x)\right) \underset{x\to 0}{=} o\left(x^3\right)$.

Ainsi,

$$\begin{aligned}
f(x) &= \ln(4)+\ln\left(1+\tfrac{x}{2}+\tfrac{x^2}{2}+\tfrac{x^3}{12}+o\left(x^3\right)\right)\\
&= \ln(4)+\tfrac{x}{2} + \tfrac{x^2}{2} + \tfrac{x^3}{12} + o\left(x^3\right)\\
&\qquad -\tfrac{1}{2}\left(\tfrac{x^2}{4} + \tfrac{x^3}{2} + o\left(x^3\right)\right)\\
&\qquad +\tfrac{1}{3}\left(\tfrac{x^3}{8} + o\left(x^3\right)\right)\\
&\qquad + o\left(x^3\right)\\
&= \ln(4)+\tfrac{x}{2} + \tfrac{x^2}{2} + \tfrac{x^3}{12} + o\left(x^3\right)\\
&\qquad -\tfrac{x^2}{8} - \tfrac{x^3}{4} + o\left(x^3\right))\\
&\qquad + \tfrac{x^3}{24} + o\left(x^3\right))\\
&\qquad + o\left(x^3\right)\\
&= \ln(4)+\tfrac{x}{2}+\tfrac{3x^2}{8}+\tfrac{2-6+1}{24}x^3+o\left(x^3\right)\\
&= \ln(4)+\tfrac{x}{2}+\tfrac{3x^2}{8}-\tfrac{x^3}{8}+o\left(x^3\right).
\end{aligned}$$

Conclusion,

$$\boxed{f(x)=\ln(4)+\frac{x}{2}+\frac{3x^2}{8}-\frac{x^3}{8}+o\left(x^3\right).}$$