

Correction de l'exercice Noël 04 Équations complexes

Solution de l'exercice 1 Puisque $2x \xrightarrow{x \rightarrow 0} 0$, on obtient que

$$\begin{aligned} 2 + \sin(2x) &\underset{x \rightarrow 0}{=} 2 + 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + o(x^5) \\ &\underset{x \rightarrow 0}{=} 2 + 2x - \frac{4x^3}{3} + \frac{8x^5}{30} + o(x^5) \\ &\underset{x \rightarrow 0}{=} 2 + 2x - \frac{4x^3}{3} + \frac{4x^5}{15} + o(x^5). \end{aligned}$$

Posons $f : x \mapsto \frac{1}{2+\sin(2x)}$. On a alors

$$f(x) \underset{x \rightarrow 0}{=} \frac{1}{2 + 2x - \frac{4x^3}{3} + \frac{4x^5}{15} + o(x^5)} \underset{x \rightarrow 0}{=} \frac{1}{2} \frac{1}{1 + x - \frac{2x^3}{3} + \frac{2x^5}{15} + o(x^5)}$$

On sait que $\frac{1}{1+u} \underset{u \rightarrow 0}{=} 1 - u + u^2 - u^3 + u^4 - u^5 + o(u^5)$. Posons $u(x) \underset{x \rightarrow 0}{=} x - \frac{2x^3}{3} + \frac{2x^5}{15} + o(x^5)$. Alors,

- $u(x) \xrightarrow{x \rightarrow 0} 0$.
- De plus,

$$\begin{aligned} u(x)^2 &\xrightarrow{x \rightarrow 0} \left(x - \frac{2x^3}{3} + \frac{2x^5}{15} + o(x^5)\right) \left(x - \frac{2x^3}{3} + \frac{2x^5}{15} + o(x^5)\right) \\ &\xrightarrow{x \rightarrow 0} x^2 - \frac{2x^4}{3} + o(x^5) \\ &\quad - \frac{2x^4}{3} + o(x^5) \\ &\quad + o(x^5) \\ &\xrightarrow{x \rightarrow 0} x^2 - \frac{4x^4}{3} + o(x^5). \end{aligned}$$

- Puis,

$$\begin{aligned} u(x)^3 &\xrightarrow{x \rightarrow 0} \left(x - \frac{2x^3}{3} + \frac{2x^5}{15} + o(x^5)\right) \left(x^2 - \frac{4x^4}{3} + o(x^5)\right) \\ &\xrightarrow{x \rightarrow 0} x^3 - \frac{4x^5}{3} + o(x^5) \\ &\quad - \frac{2x^5}{3} + o(x^5) \\ &\quad + o(x^5) \\ &\xrightarrow{x \rightarrow 0} x^3 - 2x^5 + o(x^5). \end{aligned}$$

- Et encore,

$$\begin{aligned} u(x)^4 &\xrightarrow{x \rightarrow 0} \left(x^2 - \frac{4x^4}{3} + o(x^5)\right) \left(x^2 - \frac{4x^4}{3} + o(x^5)\right) \\ &\xrightarrow{x \rightarrow 0} x^4 + o(x^5). \end{aligned}$$

- Puisque $u(x) \underset{x \rightarrow 0}{\sim} x$, alors $u(x)^5 \underset{x \rightarrow 0}{\sim} x^5$ et donc

$$u(x)^5 \xrightarrow{x \rightarrow 0} x^5 + o(x^5).$$

- Enfin, $o(u(x)^5) \underset{x \rightarrow 0}{=} o(x^5)$.

Ainsi,

$$\begin{aligned}
 f(x) &\underset{x \rightarrow 0}{=} \frac{1}{2} (1 - u(x) + u^2(x) - u^3(x) + u^4(x) - u^5(x) + o(u(x)^5)) \\
 &\underset{x \rightarrow 0}{=} \frac{1}{2} (1 - x + \frac{2x^3}{3} - \frac{2x^5}{15} + o(x^5) \\
 &\quad + x^2 - \frac{4x^4}{3} + o(x^5) \\
 &\quad - x^3 + 2x^5 + o(x^5) \\
 &\quad + x^4 + o(x^5) \\
 &\quad - x^5 + o(x^5) \\
 &\quad + o(x^5)) \\
 &\underset{x \rightarrow 0}{=} \frac{1}{2} \left(1 - x + x^2 - \frac{x^3}{3} - \frac{x^4}{3} + \frac{13x^5}{15} + o(x^5) \right).
 \end{aligned}$$

Conclusion,

$$\boxed{\frac{1}{2 + \sin(2x)} \underset{x \rightarrow 0}{=} \frac{1}{2} - \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{6} + \frac{13x^5}{30} + o(x^5).}$$

Solution de l'exercice 2

1. On a les égalités entre complexes suivantes :

$$\begin{aligned}
 1 + 3i + z^6 &= 4 \quad \Leftrightarrow \quad z^n = 3 - 3i \\
 &\Leftrightarrow \quad z^6 = 3\sqrt{2} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) \\
 &\Leftrightarrow \quad z^6 = 3\sqrt{2} e^{-i\frac{\pi}{4}} \\
 &\Leftrightarrow \quad \exists k \in \llbracket 0; 5 \rrbracket, \quad z = \sqrt[6]{3\sqrt{2}} e^{i(-\frac{\pi}{24} + \frac{2k\pi}{6})}.
 \end{aligned}$$

Conclusion, l'ensemble des solutions est

$$\boxed{\mathcal{S} = \left\{ \sqrt[6]{3\sqrt{2}} e^{i(-\frac{\pi}{24} + \frac{k\pi}{3})} \mid k \in \llbracket 0; 5 \rrbracket \right\}.}$$

2. Posons $p = n - k$ i.e. $k = n - p$. Si $k = 0$, $p = n$ et si $k = n - 1$, $p = 1$. Dès lors, on a les égalités suivantes :

$$\begin{aligned}
 \sum_{k=0}^{n-1} |a + \omega^k b| &= \sum_{p=1}^n |a + \omega^{n-p} b| \\
 &= \sum_{p=1}^n |a + \omega^n \omega^{-p} b| \\
 &= \sum_{p=1}^n |a + \omega^{-p} b| \quad \text{car } \omega^n = 1 \text{ car } \omega \in \mathbb{U}_n \\
 &= \sum_{p=1}^n |\omega^{-p}| |a \omega^p + b|.
 \end{aligned}$$

Or $\omega \in \mathbb{U}_n$ donc $\omega \in \mathbb{U}$ i.e. $|\omega| = 1$. Conclusion,

$$\boxed{\sum_{k=0}^{n-1} |a + \omega^k b| = \sum_{p=1}^n |a \omega^p + b|.}$$