$f : x \mapsto e^{\sin(x)}$

$$\sin(x) \underset{x\to 0}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5) \quad \checkmark$$

Posons $u \underset{x\to 0}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$

- $u \underset{x\to 0}{\longrightarrow} 0 \quad \checkmark$
- $e^u \underset{u\to 0}{=} 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + o(u^5)$
- $$\begin{aligned} u^2 &\underset{x\to 0}{=} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right) \\ &\underset{x\to 0}{=} x^2 - \frac{x^4}{3!} + o(x^5) \quad \checkmark \\ &\qquad - \frac{x^4}{3!} + o(x^5) \quad \times \\ &\qquad + o(x^5) \quad \checkmark \\ &\underset{x\to 0}{=} x^2 - \frac{x^4}{3} + o(x^5) \quad \text{oui} \end{aligned}$$
- $$\begin{aligned} u^3 &\underset{x\to 0}{=} \left(x^2 - \frac{x^4}{3} + o(x^5)\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right) \quad \checkmark \\ &\underset{x\to 0}{=} x^3 - \frac{x^5}{3} + o(x^5) \\ &\qquad - \frac{x^5}{3!} + o(x^5) \\ &\qquad + o(x^5) \\ &\underset{x\to 0}{=} x^3 - \frac{x^5}{2} + o(x^5) \end{aligned}$$
- $$\begin{aligned} u^4 &\underset{x\to 0}{=} \left(x^3 - \frac{x^5}{2} + o(x^5)\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right) \\ &\underset{x\to 0}{=} x^4 + o(x^5) \quad \checkmark \end{aligned}$$
- $$\begin{aligned} u^5 &\underset{x\to 0}{=} \left(x^4 + o(x^5)\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right) \\ &\underset{x\to 0}{=} x^5 + o(x^5) \quad \checkmark \end{aligned}$$
- $o(u^5) \underset{x\to 0}{=} o\left(x^5 + o(x^5)\right) \underset{x\to 0}{=} \cancel{o(o(x^5))} \underset{x\to 0}{=} o(x^5)$ Non oui

Donc,

$$\begin{aligned} f(x) = e^{\sin(x)} &\underset{x\to 0}{=} 1 + x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5) \\ &\qquad + \frac{x^2}{2} - \frac{x^4}{6} + o(x^5) \\ &\qquad + \frac{x^3}{3!} - \frac{x^5}{12} + o(x^5) \quad \checkmark \\ &\qquad + \frac{x^4}{4!} + o(x^5) \\ &\qquad + \frac{x^5}{5!} + o(x^5) + o(x^5) \\ &\underset{x\to 0}{=} 1 + x + \frac{x^2}{2} - \frac{3x^4}{24} - \frac{4x^5}{60} + o(x^5) \quad \checkmark \end{aligned}$$

Conclusion :

$$\boxed{f(x) = e^{\sin(x)} \underset{x\to 0}{=} 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + o(x^5)}$$

TB !

$A = \begin{pmatrix} 2 & 1 & 1 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}$; $B = \begin{pmatrix} 0 & -1 & -1 \\ 2 & 4 & 3 \\ -2 & -3 & -2 \end{pmatrix}$; $C = A - B = \begin{pmatrix} 2 & 2 & 2 \\ -4 & -4 & -4 \\ 4 & 4 & 4 \end{pmatrix}$ ✓

$A = C + B$

$$BC = \begin{pmatrix} 0 & -1 & -1 \\ 2 & 4 & 3 \\ -2 & -3 & -2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ -4 & -4 & -4 \\ 4 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ ✓

$$CB = \begin{pmatrix} 2 & 2 & 2 \\ -4 & -4 & -4 \\ 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 2 & 4 & 3 \\ -2 & -3 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ ✓

Ainsi, on a $BC = O_3 = CB$, i.e. les matrices B et C commutent.
Donc, d'après la formule du binôme de Newton, on a :

$$\forall n \in \mathbb{N}^*, \; A^n = (C+B)^n = \sum_{k=0}^{n} \binom{n}{k} C^k B^{n-k}$$ oui

Si $n \geqslant 2$ $$= \sum_{k=1}^{n-1} \binom{n}{k} C^k B^{n-k} + \binom{n}{0} C^0 B^n + \binom{n}{n} C^n B^0$$ ✓

De plus, B et C commutent, donc :

$$A^n = \sum_{k=1}^{n-1} \binom{n}{k} (BC) C^{k-1} B^{n-1-k} + I_3 B^n + C^n I_3$$ ✓

Or, on sait que $BC = O_3$, donc :

$$A^n = O_3 + B^n + C^n$$ Oui Faire $n = 1$ à part.

$\boxed{\text{Conclusion : } \forall n \in \mathbb{N}^*, \; A^n = B^n + C^n}$ bien.