Lilian
OBR/07

On a $\frac{x}{x+1} \underset{x \to 0}{=} x - x^2 + x^3 + o(x^3)$ ✓

Donc $ch\left(\frac{x}{x+1}\right) \underset{x \to 0}{=} ch\left(x - x^2 + x^3 + o(x^3)\right)$

On pose $u = x - x^2 + x^3 + o(x^3) \xrightarrow[x \to 0]{} 0$ Oui

Or $ch(u) \underset{u \to 0}{=} 1 + \frac{u^2}{2} + o(u^3)$ ✓

$u^2 = \left(x - x^2 + x^3 + o(x^3)\right)^2$
$= x^2 - x^3 + o(x^3)$
$\quad - x^3 + o(x^3)$
$\quad + o(x^3)$
$= x^2 - 2x^3 + o(x^3)$ ✓ Oui

$u^3 = x^3 + o(x^3)$ ✓

$o(u^3) = o(x^3)$ ✓

$$\boxed{ch\left(\frac{x}{1+x}\right) \underset{x \to 0}{=} 1 + \frac{x^2}{2} - x^3 + o(x^3)}$$ TB!

Exercice 2

1) On a

$$P(A) = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} - 5\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -5 \\ -10 & 11 \end{pmatrix} - \begin{pmatrix} 10 & -5 \\ -10 & 15 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ ✓

Donc $\boxed{P(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$ ✓

2) On a

$A^2 - 5A + 4I_2 = 0 \iff A^2 - 5A = -4I_2$ ✓

$\iff \frac{5A - A^2}{4} = I_2$ ✓

$\iff A\,\frac{5I_2 - A}{4} = I_2$ ✓

A est inversible et

~~On en~~ Donc $\boxed{A^{-1} = \frac{5I_2 - A}{4}}$ Oui

3) Si $n \leq 1$ on a

$R_n = X^2 - 5X + 4$

Si $n \geq 2$ ~~on a~~ on a

$\deg(X^n) > \deg(X^2 - 5X + 4)$

Donc $\exists\, Q(X)$, $\deg(Q(X)) = n - 2$
et $\exists\, R_n(X)$, $\deg(R_n(X)) \leq 1$

donc $R_n(X) = aX + b$ avec $(a,b) \in \mathbb{R}^2$ Oui !

On a
$X^n = Q(X)(X^2 - \cancel{4}5X + 4) + R_n(X)$ ✓
$= Q(X)(X^2 - 5X + 4) + aX + b$

Soit Δ le déterminant associé à $X^2 - 5X + 4$ on a
$\Delta = 25 - 16 = 9 > 0$
Donc $X_1 = \frac{5-3}{2} = 1$ $X_2 = \frac{5+3}{2} = 4$ ✓

Si $X = 1$ on a
$1 = a + b$ ✓

Si $X = 4$ on a
$4^n = 4a + b$ ✓

On a

$\begin{cases} 1 = a + b \\ 4^n = 4a + b \end{cases} \iff 4^n - 1 = 3a \iff a = \frac{4^n - 1}{3}$ ✓

On a $1 = \frac{4^n - 1}{3} + b \iff b = 1 + \frac{1 - 4^n}{3} \iff b = \frac{4 - 4^n}{3}$ ✓

Donc $\boxed{\begin{array}{l} \text{si } n \geqslant 2 \quad R_n = \dfrac{4^n-1}{3} X + \dfrac{4-4^n}{3} \\ \text{si } n \leqslant 1 \quad R_n = X^2 - 5X + 4 \end{array}}$ TB

4) On a

$$A^n = Q(A)(A^2 - 5A + 4I_2) + \frac{4^n-1}{3} A + \frac{4-4^n}{3} I_2$$

$$= \frac{4^n-1}{3} A + \frac{4-4^n}{3} I_2$$

$$= \frac{4^n-1}{3} \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} + \frac{4-4^n}{3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\frac{4^n-1}{3} & \frac{-4^n+1}{3} \\ -2\frac{4^n-1}{3} & \frac{4^n-1}{3} \text{ x3} \end{pmatrix} + \begin{pmatrix} \frac{4-4^n}{3} & 0 \\ 0 & \frac{4-4^n}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\cdot 4^n - 4^n - 2 + 4}{3} & \frac{1-4^n}{3} \\ -2\frac{4^n-1}{3} & \cancel{\frac{3}{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4^n+2}{3} & \frac{1-4^n}{3} \\ -2\frac{4^n-1}{3} & \cancel{1} \end{pmatrix}$$

Bien !

Donc $\boxed{A^n = \begin{pmatrix} \frac{4^n+2}{3} & \frac{1-4^n}{3} \\ -2\frac{4^n-1}{3} & 1 \end{pmatrix}}$